Deciding active structural completeness

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LATD
Bern, August 2018

(active) structural completeness

$\vdash_r$ - the least consequence relation containing the rule r and extending a cr $\vdash$

r is <u>admissible for</u> $\vdash$ if Theorems($\vdash$) = Theorems($\vdash_r$)

$r = \Gamma/\varphi$ is <u>active</u> for $\vdash$ if there is a substitusion σ such that $\sigma(\Gamma) \subseteq$ Theorems($\vdash$)

$\vdash$ is <u>(actively) structurally complete</u> if every (active) admissible rule is derivable, i.e., is in $\vdash$

Fact

Γ/φ is admissible for $\vdash$ iff $(\forall\gamma \in \Gamma, \vdash \sigma(\gamma))$ yields $\vdash \sigma(\varphi)$ for every substitution σ

(quasi)varieties

identities look like $(\forall\bar{x})\ s(\bar{x}) \approx t(\bar{x})$
quasi-identities look like

$$(\forall\bar{x})\ s_1(\bar{x}) \approx t_1(\bar{x}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \ \rightarrow\ s(\bar{x}) \approx t(\bar{x})$$

(quasi)varieties = classes of algebras defined by (quasi-)identities

Mal'cev
A class is $\mathrm{SPP_U}$-closed iff it is a quasivariety.

Birkhoff
A class is HSP-closed iff it is a variety.

correspondence

cr $\vdash$	$\leftrightsquigarrow$	quasivariety $\mathcal{Q}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	identities valid in $\mathcal{Q}$
Theorems($\vdash$)	$\leftrightsquigarrow$	free algebra $\mathbf{F}$
derived rules	$\leftrightsquigarrow$	quasi-identities valid in $\mathcal{Q}$
admissible rules	$\leftrightsquigarrow$	quasi-identities valid in $\mathbf{F}$
active rules	$\leftrightsquigarrow$	quasi-identities with the premise satisfiable in $\mathbf{F}$

Thus we study admissibility and (A)SC for (quasi)varieties.

SC and AS, a comparition

Examples

- S5 and $Ł_n$ are ASC but not SC ($n \geqslant 3$) [folklore];
- discriminator varieties are ASC [Burris '92, Dzik '11], and are SC iff they are minimal or trivial (if there are two distinct constants) [Campercholi, S., Vaggione '16];
- ASC normal extensions of S4 are SC iff they extend S4.McKinsey [Dzik and S. '16];
- among 3330 3-element groupoids (up to izo.) 2676 generate SC quasivarieties and 2930 generate ASC quasivarieties [Metcalfe and Röthlisberger '13];
- almost all finite algebras generate SC varieties [Murskiĭ '75].
- among 97 224 120 normal modal logics given by frames up to 6 elements there are 5 066 204 SC and 73 664 964 ASC (and 14 uncounted) [S. and Uliński unpublished];

decidability

(A)SC-problem for quasivarieties

INPUT: a finite set of finite algebras $\mathcal{K}$,
OUTPUT: YES if $\mathrm{SP}(\mathcal{K})$ is (A)SC, NO otherwise.

Th. (Dywan '78, Bergman '88, Metcalfe & Röthlisberger '13)

There is an algorithm which solves the (A)SC-problem for quasivarieties.

- Dywan reduced the number of quasi-identities to be checked.
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(A)SC-problem for varieties

INPUT: a finite algebra $\mathbf{A}$,
OUTPUT: YES if $HSP(\mathbf{A})$ is (A)SC, NO otherwise.

Question

How about the (A)SC-problem for varieties?

SI algebras

An algebra $\mathbf{A}$ is subdirectly irreducible if there is a pair $a, b \in A$ of distinct elements such that every nontrivial congruence of $\mathbf{A}$ contains (a, b).

Fact

An algebra $\mathbf{A}$ is SI if and only if whenever $\mathbf{A} \leqslant \prod \mathbf{A}_i$, then one of the projections $\pi_i : \mathbf{A} \to \mathbf{A}_i$ is an embeding.

A quasivariety $\mathcal{Q}$ is finitely generated if there exists a finite family $\mathcal{F}$ of finite algebras such that $\mathcal{Q} = \mathrm{SP}(\mathcal{F})$.

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A quasivariety $\mathcal{Q}$ is finitely generated if there exists a finite family $\mathcal{F}$ of finite algebras such that $\mathcal{Q} = \mathsf{SP}(\mathcal{F})$.

Theorem (Birkhoff '35)

Every variety $\mathcal{V}$ is generated as a quasivariety by its SI algebras:

$$\mathcal{V} = \mathsf{SP}(\text{SI algebras from } \mathcal{V}).$$

Consequently, there is a finite bound on the size of SI algebras in $\mathcal{V}$ if and only if $\mathcal{V}$ is finitely generated as a quasivariety.

Question

How about the (A)SC-problem for varieties?

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Proposition

There is an algorithm which solves the (A)SC-problem restricted to congruence distributive varieties.

Proof.

By Jónsson's lemma, a finitely generated congruence distributive variety is finitely generated as a quasivariety.
All SI algebras from $\mathsf{HSP}(\mathbf{A})$ are in $\mathsf{HS}(\mathbf{A})$. □

Application

All varieties with lattice terms are congruence distributive.

SI algebras in SC varieties

Theorem (Bergman '88)

Let $\mathbf{A}$ be a finite algebra. If $\mathsf{HSP}(\mathbf{A})$ is SC, then every SI algebra from $\mathsf{HSP}(\mathbf{A})$ embeds into $\mathbf{A}$.
Consequently, a finitely generated SC variety is finitely generated as a quasivariety.

SI algebras in ASC varieties - the new result!

Theorem

Let $\mathbf{A}$ be a finite algebra. If HSP($\mathbf{A}$) is ASC, then every SI algebra from HSP($\mathbf{A}$) has cardinality bounded by

$$|A|^{(|A|+1)\cdot|A|^{2\cdot|A|}}$$

Consequently, a finitely generated SC variety is finitely generated as a quasivariety.

Proof.

By studying relatively SI algebras in the quasivariety generated by free algebras. □

The residual problem

Theorem (McKenzie '96)

There is no algorithm which takes as an input a finite algebra $\mathbf{A}$ and decides whether the cardinalities of SI algebras in $\mathrm{HSP}(\mathbf{A})$ are bounded by a finite number.

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The residual problem

INPUT: a finite algebra $\mathbf{A}$ and a natural number m,
OUTPUT: YES if all SI algebras from HSP($\mathbf{A}$) have cardinality $\leqslant m$, NO otherwise.

We do not know whether there exists an algorithm solving the residual problem!

Application

Fact

Let $\mathcal{A}$ be a set of finite algebras. Assume that there is an algorithm solving the residual problem restricted to the case when the input algebras are from $\mathcal{A}$. Then there is an algorithm solving the (A)SC problem for varieties restricted to algebras from $\mathcal{A}$.

Proof.

Take $m = |A|^{(|A|+1)\cdot|A|^{2\cdot|A|}}$. $\square$

Application

Theorem

There are algorithms solving the residual problem for the following classes of algebras:

- finite semigroups [Golubov and Sapir '82, McKenzie '83, Kublanovskiĭ '83];
- finite algebras generating congruence modular varieties [Freese and McKenzie '81]
  includes varieties with Mal'cev's term (groups, modules)
  $M(x, y, y) = M(y, y, x) = x$;
- finite algebras generating congruence meet-semidistributive varieties [Willard '00]
  includes varieties with a semilattice term.

Consequently, there are algorithms solving the (A)SC problem for varieties when restricted to one of the above classes of algebras.

The end

This is all Thank you!